Remark 6.1. If the contour Γ in relation (6.6) has been chosen as in Example 2, then the corresponding infinite system is of exactly the same form as system (2.19) of [4]. This system can be investigated by the method proposed in [5].

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ASYMPTOTIC SOLUTION OF THE CONTACT PROBLEM FOR A THIN ELASTIC LAYER

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The contact problem of impressing a stamp in an elastic layer of finite thickness h lying without friction or adhering rigidly to an undeformable foundation is considered. The frictional forces between the stamp and the surface layer are assumed absent, and the surface layer outside the stamp is not loaded. The contact domain Ω between the stamp and the layer is assumed simply connected (*) and fixed.

An asymptotic solution of the above-mentioned problem has been obtained in [1-3] under the assumption that the relative thickness of the layer is sufficiently large, i.e. the dimensionless parameter $\lambda = h / a$, $a = {}^1/_2 \max R_{PQ}$ for any P and $Q \in \Omega$, is large.

A scheme for constructing the asymptotic solution of the mentioned problem under the assumption that the relative thickness of the layer is small has been expounded in [4].

*) Simple connectedness is assumed just for simplicity. The asymptotic method expounded below for the solution can be utilized even in the case of a multiply connected domain Ω.

A more general and convenient scheme for the construction of the asymptotic solution of the problem for small relative thickness of the layer is given herein on the basis of the main idea in [4]. The results are presented in the form of simple formulas suitable for practical utilization.

1. Basic integral equation of the problem. Introduction of a dimensionless parameter characterizing the smallness of the relative layer thickness. As is known [1-3], the above-mentioned problem is reduced, by operational calculus methods, to the solution of the integral equation

$$\iint\limits_{\Omega} q(P)\, K\left(\frac{R}{h}\right) d\Omega = 2\pi h \Delta \delta(Q), \qquad Q \in \Omega$$
$$\Delta = G(1-\nu)^{-1}, \qquad R = \sqrt{(\xi - x)^2 + (\eta - y)^2} \tag{1.1}$$

in the distribution function of the contact pressures $q(P)$. The function $\delta(Q)$ is the settlement of the surface layer points under the stamp, and is determined by the shape of the stamp base and the degree of its indentation into the layer; G and ν are elastic constants of the layer; (ξ, η) are coordinates of the point P; (x, y) are coordinates of the point Q.

The kernel of the integral equation (1.1) has the form

$$K(t) = \int_0^\infty L(u) J_0(ut)\, du \tag{1.2}$$

where $J_0(x)$ is the Bessel function, and the function $L(u)$ is

$$\text{(a)} \qquad L(u) = \frac{\operatorname{ch} 2u - 1}{\operatorname{sh} 2u + 2u}$$
$$\text{(b)} \qquad L(u) = \frac{2\varkappa \operatorname{sh} 2u - 4u}{2\varkappa \operatorname{ch} 2u + 1 + \varkappa^2 + 4u^2}, \qquad \varkappa = 3 - 4\nu \tag{1.3}$$

Here and henceforth, cases (a) and (b) correspond to cases of a layer lying without friction or adhering rigidly to an undeformable foundation.

Furthermore, we shall assume that

1) The function $\delta(Q)$ is sufficiently smooth in Ω;

2) The solution $q(P)$ of the integral equation (1.1) exists for problems (a) and (b) in the class $L(\Omega)$ (the class of functions absolutely summable over the domain), and is unique;

3) The shrinkage of points of the surface layer $\gamma(Q)$ outside the domain of contact Ω belongs to $L(\Omega^*)$, where Ω^* is the complement of Ω in the whole plane.

For physically real cases of the problems under consideration the function $\delta(Q)$ should be strictly positive in Ω and such that $q(P) \geqslant 0$ in Ω.

Let us write down some general properties of a function $L(u)$ of the form (1.3).

The functions $L(z)$ will be odd and meromorphic (the ratio of two quasipolynomials) in the complex $z = u + iv$ plane; the functions $L(z)$ are real on the $v = 0$ axis, have a unique single zero $u = 0$, and no poles. A countable set of their complex zeros z_n and poles ζ_n is located on four branches disposed symmetrically relative to the real and imaginary axes. All the z_n and ζ_n are distinct (*), their absolute

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*) The zeros of the functions $L(z)$ for problem (a) are multiple and pure imaginary, i.e. the right and left branches merge into one of the imaginary axes.

values increase as n increases. For large n the following simple asymptotic formulas hold

$$z_n = c_1 \ln n + ic_2 n, \quad \zeta_n = d_1 \ln n + id_2 n \quad (c_1, c_2, d_1, d_2 = \text{const}) \tag{1.4}$$

Taking account of all the above, we can represent $L(z)$ of the form (1.3) as

$$L(z) = z \prod_{n=1}^{\infty} \frac{(z^2 + \delta_n^2)}{(z^2 + \gamma_n^2)}, \qquad \lim_{z \to 0} L(z) z^{-1} = A = \prod_{n=1}^{\infty} \left(\frac{\delta_n}{\gamma_n}\right)^2 \tag{1.5}$$

$$\delta_n = -iz_n, \qquad \gamma_n = -i\zeta_n$$

Here z_n and ζ_n are, respectively, the zeros and poles in the $v > 0$ half-plane. It is easy to show that for large values of u the functions $L(u)$ of the form (1.3) behave as follows:

$$L(u) = 1 + O(e^{-2u}) \tag{1.6}$$

Moreover, it can be shown that the estimate

$$L(z) = O(1) \quad \text{for} \quad k \to \infty \tag{1.7}$$

holds in the complex z -plane on any regular [5] system of contours C_k if $|\arg z| \leqslant {}^1/_2\pi - \varepsilon$ and $|\arg z - \pi| \leqslant {}^1/_2\pi - \varepsilon$; on the imaginary axis $(u=0)$

$$L(z) = O(z^{-1}) \quad \text{for} \ |z| \to \infty \tag{1.8}$$

Utilizing (1.5), (1.7) and (1.8) we can represent the meromorphic functions $L(z)$ of the form (1.3) as the sum of their principal parts [5]

$$L(u) = \frac{2u}{\pi} \sum_{m=1}^{\infty} \frac{\gamma_m s_m}{u^2 + \gamma_m^2}, \quad \lim_{u \to 0} L(u) u^{-1} = A = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{s_m}{\gamma_m} \tag{1.9}$$

It can be shown that the series (1.9) converges uniformly for all $0 \leqslant u \leqslant D < \infty$. The constants s_m have the form

$$s_m = \pi i \left\{ \left[\frac{\zeta_m}{L(\zeta_m)} \right]' \right\}^{-1} \qquad (s_m \sim \zeta_m^{-1} \quad \text{for} \quad m \to \infty) \tag{1.10}$$

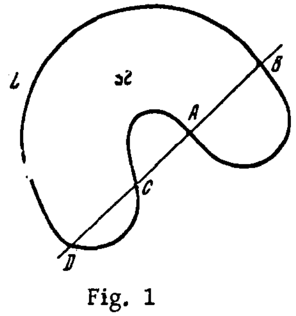

Fig. 1

Let us now turn to the question of the dimensionless parameter characterizing the smallness of the relative thickness of the layer.

Let us note that the dimensionless parameter λ introduced in [1—3] is suitable only for the characteristics of a layer of large relative thickness, because the smallness of the relative layer thickness does not follow from the smallness of this parameter in the general case. Hence, the necessity to introduce yet another dimensionless geometric parameter results.

At the point A of the contour L we draw a normal (Fig. 1); it intersects the contour L in a number of points B, C, D. Let us measure the length of the normal segments between the point A and the points of intersection. We let $2a_0$ denote the smallest possible of the lengths of the mentioned normal segments for any points $A \in L$.

Now let us introduce a new parameter μ by means of the relationship

$$\mu = h\,[\mathrm{Inf}\,(a_0, \rho_0)]^{-1} \tag{1.11}$$

where ρ_0 is the minimal radius of curvature of the contour L. Let us note that always $a_0 \geqslant \rho_0$ for a convex domain Ω.

The parameter μ introduced is suitable only for the characteristics of a layer of relatively small thickness.

Let us note that always $\mu \geqslant \lambda$. The equality holds only when the domain Ω is a circle.

2. Properties of the kernel of the integral (1.1). Nature of its solution. Let us mention some properties of the kernel $K(t)$ of the form (1.2) which are common for both problems under consideration. Using the integral

$$\int_0^\infty J_0(ut)\,du = \frac{1}{t} \tag{2.1}$$

we represent (1.2) as

$$K(t) = t^{-1} - F(t)$$
$$F(t) = \int_0^\infty [1 - L(u)]\,J_0(ut)\,du \tag{2.2}$$

On the basis of the property (1.6) of the function $L(u)$ presented above, it is easy to show that a function $F(t)$, even in t, will be continuous and continuously differentiable any number of times for $0 \leqslant t < \infty$.

If the relationship (1.9) and the integral

$$\int_0^\infty \frac{uJ_0(ua)}{u^2 + b^2}\,du = K_0(ab) \tag{2.3}$$

are used, then another representation for the kernel $K(t)$ can be obtained

$$K(t) = \frac{2}{\pi} \sum_{m=1}^{\infty} \gamma_m s_m K_0(\gamma_m t) \tag{2.4}$$

It is not difficult to prove that the series (2.4) converges uniformly and absolutely for all $0 < \varepsilon \leqslant t \leqslant \infty$. To do this it is sufficient to utilize the relationships (1.4) and (1.11).

Now let us turn to the question of the nature of the solution of (1.1). Let us rewrite it on the basis of (2.2) as

$$\iint_\Omega q(P)\,\frac{d\Omega}{R} = 2\pi\Lambda\delta(Q) + \frac{1}{h}\iint_\Omega q(P)\,F\left(\frac{R}{h}\right)d\Omega, \qquad Q \in \Omega \tag{2.5}$$

By assumption, the solution of the integral equation (1.1) or (2.5) exists in $L\ (\Omega)$. Then the function

$$\Phi(Q)=\iint_{\Omega} q\,(P)\,F\left(\frac{R}{h}\right)d\Omega \tag{2.6}$$

will be continuous and continuously differentiable with respect to x and y any number of times in Ω by virtue of the mentioned properties of $F\ (t)$.

The following results hence follow:

1) For $\lambda\to\infty$ the integral equation (2.5) goes over into the known integral equation of the corresponding contact problem for an elastic half-space

$$\iint_{\Omega} q\,(P)\,\frac{d\Omega}{R}=2\pi\Delta\delta\,(Q),\qquad Q\in\Omega \tag{2.7}$$

2) The solution of the integral equation (2.5) or (1.1) of the contact problem for a layer will be of the same nature as the solution of the integral equation (2.7) for a half-space for all nonzero values of the dimensionless parameters λ and μ :

3) If the solution of the integral equation (2.7) is known for given Ω and $\delta\ (Q)$, then to find the approximate solution of the integral equations (2.5) or (1.1) for large λ it is sufficient to approximate the function $F\ (t)$ of the form (2.2) by a degenerate function; intrinsically speaking, this idea has been used in [1-3] to obtain asymptotic solutions of (1.1) for large λ .

3. Inner solution of the integral equation (1.1) for small μ

Instead of the integral equation (1.1) let us consider its equivalent functional equation

$$Q\,(\alpha,\ \beta)\,L\,(\gamma h)\,\gamma^{-1}=\Delta W\,(\alpha,\ \beta),\qquad Q\,(\alpha,\ \beta)=\frac{1}{2\pi}\iint_{\Omega} q\,(P)\,e^{i(\alpha\xi+\beta\eta)}d\Omega \tag{3.1}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q\,(\alpha,\ \beta)\,e^{-i(\alpha x+\beta y)}d\alpha d\beta=\begin{cases} q\,(Q) & \text{in } \Omega\\ 0 & \text{in } \Omega^*\end{cases} \tag{3.2}$$

$$W\,(\alpha,\ \beta)=-\frac{1}{2\pi}\iint_{\Omega}\delta\,(P)\,e^{i(\alpha\xi+\beta\eta)}d\Omega-\frac{1}{2\pi}\iint_{\Omega^*}\gamma\,(P)\,e^{i(\alpha\xi+\beta\eta)}d\Omega$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W\,(\alpha,\ \beta)\,e^{-i(\alpha x+\beta y)}d\alpha d\beta=\begin{cases} -\delta\,(Q) & \text{in } \Omega\\ -\gamma\,(Q) & \text{in } \Omega^*\end{cases}$$

On the basis of (3.1), (3.2), we obtain

$$q\,(Q)=\frac{\Delta}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\gamma W\,(\alpha,\ \beta)}{L\,(\gamma h)}\,e^{-i(\alpha x+\beta y)}d\alpha d\beta,\qquad Q\in\Omega \tag{3.3}$$

or finally

$$q\,(Q)=\frac{\Delta}{2\pi h^2}\left[\iint_{\Omega}\delta\,(P)\,M\left(\frac{R}{h}\right)d\Omega+\iint_{\Omega^*}\gamma\,(P)\,M\left(\frac{R}{h}\right)d\Omega\right]\qquad Q\in\Omega \tag{3.4}$$

$$M(t) = \int_0^\infty \frac{u^2}{L(u)} J_0(ut)\,du \tag{3.5}$$

For a function $M(t)$ of the form (3.5) it is possible to obtain the expansion

$$M(t) = \frac{2}{\pi} \sum_{m=1}^{\infty} \delta_m v_m K_0(\delta_m t) \tag{3.6}$$

just as has been done in Sect. 2 for the kernel $K(t)$. As before, it can be shown that the series (3.6) converges absolutely and uniformly for all $0 < \varepsilon \leqslant t \leqslant \infty$.

Let us consider the geometric locus of points $Q \in \Omega$ not less than $a_0\,\varepsilon$ removed from the boundary of the domain Ω along the normal (see Sect. 1 for definition of the quantity a_0). These points evidently occupy a certain domain $\Omega_\varepsilon \subset \Omega$.

Now, on the basis of (3.6) we easily obtain the following estimate for the second integral in (3.4) for all points $Q \in \Omega_\varepsilon$ for small values of the parameter μ:

$$\left|\iint_{\Omega^*} \gamma(P) M\left(\frac{R}{h}\right) d\Omega\right| = O\left[\left(\frac{\varepsilon}{\mu}\right)^{-1/2} \exp\left(-\frac{\varepsilon}{\mu}\operatorname{Re}\delta_1\right)\right] \tag{3.7}$$

In deducing the estimate it has also been taken into account that $\gamma(Q) \in L(\Omega^*)$. Therefore, for small μ the following asymptotic relationship holds

$$q(Q) = \frac{\Delta}{2\pi h^3} \iint_{\Omega} \delta(P) M\left(\frac{R}{h}\right) d\Omega + O\left[\left(\frac{\varepsilon}{\mu}\right)^{-1/2} \exp\left(-\frac{\varepsilon}{\mu}\operatorname{Re}\delta_1\right)\right], \quad Q \in \Omega_\varepsilon \tag{3.8}$$

Let us designate it the inner solution of the integral equation (1.1) for small μ.

Let us continue the function $\delta(Q)$ in the domain Ω^*. For example, let $\psi(Q) = \delta(Q)$ in Ω and $\psi(Q) = \delta^*(Q)$ in Ω^*. Let us demand only that the function $\psi(Q)$ be absolutely integrable in the whole plane. Then the asymptotic equality (3.8) can be rewritten in a form more convenient for practical utilization

$$q(Q) = \frac{\Delta}{2\pi h^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(P) M\left(\frac{R}{h}\right) d\Omega + O\left[\left(\frac{\varepsilon}{\mu}\right)^{-1/2} \exp\left(-\frac{\varepsilon}{\mu}\operatorname{Re}\delta_1\right)\right], \quad Q \in \Omega_\varepsilon \tag{3.9}$$

As an illustration, let us consider the case $\delta(Q) \equiv \delta$ for $Q \in \Omega$ (a flat stamp). Let us describe a circle of radius $a = {}^1/_2 \max_\Omega R$ around the domain Ω and let us take the following as the function $\psi(Q)$:

$$\psi(Q) = \delta \quad \text{for } r < a, \quad \psi(Q) = 0 \quad \text{for } r > a \quad (r = \sqrt{x^2 + y^2}) \tag{3.10}$$

Then after a number of manipulations, we obtain the following inner solution by means of (3.9):

$$q(Q) = \frac{\Delta\delta}{Ah} + O\left[\exp\left(-\frac{\varepsilon}{\mu}\operatorname{Re}\delta_1\right)\right], \quad Q \in \Omega_\varepsilon \tag{3.11}$$

Let us note that (3.11) agrees with the degenerate solution for small relative layer thickness given in [3] to the accuracy of the expotential component.

Analogously, for the case $\delta(Q) = \gamma r^2/a^2$ with $Q \in \Omega$ (parabolic stamp), we obtain the inner solution as

$$q(Q) = \frac{\Delta\gamma}{Ah}\left(\frac{r^2}{a^2} - 4D_1\lambda^2\right) + O\left[\exp\left(-\frac{\varepsilon}{\mu}\operatorname{Re}\delta_1\right)\right], \quad Q \in \Omega_\varepsilon \tag{3.12}$$

The constant D_1 in (3.12) is

$$D_1 = \frac{A}{2}\lim \frac{d^2}{du^2}\left(\frac{u}{L(u)}\right) \quad \text{for} \quad u \to 0 \tag{3.13}$$

The constant A in (3.11) - (3.13) is defined by the relationship (1.5).

4. Construction of a boundary-layer type solution of the integral equation (1.1) for small μ in the domain $\Omega - \Omega_\varepsilon$

Let us rewrite the integral equation (1.1) as

$$\iint\limits_{\Omega-\Omega_{\varepsilon_1}} q(P)K\left(\frac{R}{h}\right)d\Omega + \iint\limits_{\Omega_{\varepsilon_1}} q(P)K\left(\frac{R}{h}\right)d\Omega = 2\pi h\Delta\delta(Q) \tag{4.1}$$

$$Q \in \Omega, \quad 1 > \varepsilon_1 > \varepsilon$$

Now, if we limit ourselves to an examination of the domain $Q \in \Omega - \Omega_\varepsilon$ and take into account that the function $q(Q) \in L(\Omega_{\varepsilon_1})$, then we will have on the basis of (4.1)

$$\iint\limits_{\Omega-\Omega_{\varepsilon_1}} q(P)K\left(\frac{R}{h}\right)d\Omega + O\left[\sqrt{\mu}\exp\left(-\frac{\varepsilon_1-\varepsilon}{\mu}\operatorname{Re}\gamma_1\right)\right] = 2\pi h\Delta\delta(Q) \tag{4.2}$$

$$Q \in \Omega - \Omega_\varepsilon, \quad 1 > \varepsilon_1 > \varepsilon$$

In deducing (4.2) the relationship (2.4) was also taken into account.

Thus, it is necessary to find the solution of the integral equation (4.2) in the domain $\Omega - \Omega_{\varepsilon_1}$, where it should evidently have boundary-layer form, i.e. have a singularity [6] of the form $R^{-1/2}$ on the contour L, and tend rapidly to the inner solution (3.8) upon receding deep into the domain from the contour L.

To construct this boundary layer solution, let us transform to new variables connected to the contour L in (4.2).

From the point $A(x, y) \in \Omega - \Omega_\varepsilon$ let us drop a normal on the contour L; let the length of this normal be n, and the point of its intersection with the contour $B(x_1, y_1)$. On the contour L let us select some point $O(x_0, y_0)$ as origin, and let us measure the distance s between the points O and B along the contour L. We take the quantities n and s as the new coordinates of the point A in the curvilinear (n, s) coordinate system. Under the conditions $\varepsilon < 1$ and $-l/2 < s < l/2$ (l is the perimeter of the contour L), only one pair of numbers (n, s) will correspond to each pair of numbers (x, y) in the domain $\Omega - \Omega_\varepsilon$, and conversely.

The integral equation (4.2) becomes in the (n, s) coordinate system

$$\int\limits_0^{\varepsilon_1/\mu} d\beta \int\limits_{-k/\mu}^{k/\mu} \varphi(\beta, \gamma)K(R_0)d\gamma + O\left[\mu^{-1/2}\exp\left(-\frac{\varepsilon_1-\varepsilon}{\mu}\operatorname{Re}\gamma_1\right)\right] = \frac{2\pi\Delta}{h}f(b, c) \tag{4.3}$$

$$0 < b < \varepsilon/\mu, \quad |c| < k/\mu, \quad 1 > \varepsilon_1 > \varepsilon$$

Here

$$b = n / h, \qquad c = s / h, \qquad \beta = \nu / h, \qquad \gamma = \tau / h, \qquad R_0 = R / h,$$

$$k = l / 2a_0 \qquad \varphi(\beta, \gamma) \equiv q(P), \qquad f(b, c) \equiv \delta(Q) \tag{4.4}$$

Furthermore, for simplicity of the exposition we limit ourselves to the consideration of an important particular case (*)

$$f(b,c) \equiv f(b) \tag{4.5}$$

It includes the case of axial symmetry as well as all cases of stamps with flat bases ($\delta(Q) \equiv \text{const}$).

To find the principal term of the asymptotic of the solution of the integral equation (4.3) for small μ we let μ^{-1} tend to infinity in (4.3) and we find the solution of the integral equation thus obtained. Let us note that letting μ^{-1} tend to infinity is equivalent to rectification of the contour L into an infinite line in the (n, s) coordinate system (**). It is hence easy to see that the belt domain $0 \leqslant b \leqslant \varepsilon / \mu$, $|c| \leqslant k / \mu$ is expanded into a half-plane, the (n, s) coordinate system degenerates into a rectangular system, the function $\varphi(\beta; \gamma)$ turns out to be dependent only on the variable β, i.e. $\varphi(\beta, \gamma) \equiv \varphi(\beta)$.

Taking all this into account, after integrating the inner integral with respect to γ we represent the integral equation (4.3) as $\mu \to 0$ as

$$\int_0^\infty \varphi(\beta) N(\beta - b) d\beta = \frac{\pi\Delta f(b)}{h}, \qquad 0 < b < \infty \tag{4.6}$$

$$N(t) = \int_0^\infty \frac{L(u)}{u} \cos ut du \tag{4.7}$$

Equation (4.6) is the integral equation of the problem of the effect of a semi-infinite stamp on an elastic strip of finite thickness h in [7]. Therefore, the principal term of the asymptotic of the solution of integral equation (4.2) or (1.1) in the domain $\Omega - \Omega_\varepsilon$ for small μ and condition (4.5) will be a plane boundary layer, determined from the

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*) The case

$$f(b, c) = \sum_{n=0}^{N} f_n(b) \cos \zeta_n c \qquad \left(\zeta_n = \frac{r\pi n\mu}{k}\right)$$

can also be studied with slight complications.

**) It follows from the above that the principal term of the asymptotic of the solution of the integral equation (4.3) for small μ will be independent of the curvature of the contour L at any of its points. However, the influence of the curvature could be taken into account by paying attention to the fact that the solution of (4.3) in the neighborhood of each point of the contour L with coordinate c and radius of curvature $\rho(c)$ agrees asymptotically for small μ with the solution of the corresponding integral equation for a stamp of circular planform with radius $\rho(c)$ if $\rho(c) > 0$, and for a stamp which is the exterior of a circle of radius $\rho(c)$ if $\rho(c) < 0$. It can thereby be established that the correction to the curvature is of the order of $O[\lambda^{2}(c)]$ on the contour L, where $\lambda(c) = h[\rho(c)]^{-1}$

Wiener-Hopf integral equation (4.6), (4.7).

Let us introduce yet another boundary layer into the considerations, and let us designate it as "outer" in contrast to the previous "inner".

The presence of an outer boundary layer can be established in studying the function $\gamma(Q)$, the distribution of surface layer shrinkage in the domain Ω^*, i.e. outside the stamp. Indeed, the function $\gamma(Q)$ takes on some finite values on the contour L of the domain Ω and tends rapidly to zero upon receding from the contour L, and the more rapidly the smaller the relative layer thickness μ.

If a scheme and reasoning analogous to that utilized above in constructing the inner boundary layer are applied to determining the outer boundary layer, and if we limit ourselves to the same accuracy, then it turns out that the principal part of the outer boundary layer is also plane and given by the relationship

$$\begin{gathered}\gamma(Q)\sim g(b) \quad \text{при} \quad Q\in\Omega^*-\Omega_\varepsilon^*,\quad \mu\to 0\\ g(b)=\frac{h}{\pi\Delta}\int_0^\infty \varphi(\beta)N(\beta-b)\,d\beta,\quad -\infty<b<0\end{gathered} \tag{4.8}$$

Here $N(t)$ also has the form of (4.7), the domain Ω_ε^* is obtained by eliminating all points distant not less than $a_0\varepsilon$, $0<\varepsilon<1$ from the contour L along the normal, from the domain Ω^*.

Let us note that $g(b)$ can be found uniquely from $\varphi(\beta)$ in solving the Wiener-Hopf integral equation (4.6).

Let us turn to a study of the question of the limits of applicability of the asymptotic equality established above for small μ.

$$q(P)\sim\varphi(\beta) \quad \text{при} \quad P\in\Omega-\Omega_\varepsilon \tag{4.9}$$

To do this let us first note what has been principally at the basis of the derivation.

Namely, the completely obvious physical fact has been utilized that asymptotically for small μ a state of stress in the neighborhood of any one point of the contour L of the domain Ω should not influence the state of stress in the neighborhood of any other point of the contour L, i.e. asymptotically there should be no mutual interaction between different points of the contour L. But this mutual influence can be internal (through the domain Ω) and external (through the domain Ω^*). The value of studying the outer boundary layer hence becomes obvious at once.

It is now clear that it is only possible to assess the limits of applicability of the asymptotic equality (4.9) by the damping rates of the inner and outer boundary layers, i.e. in other words, by their "thickness".

Let us define the relative boundary layer thickness H_1, H_2 by means of the expressions

$$\begin{gathered}H_1=b_1/h,\quad |\varphi(b_1)-\varphi^*(b_1)|\ [\varphi^*(b_1)]^{-1}=0.025\\ H_2=b_2/h,\quad g(b_2)\ [g(0)]^{-1}=0.025\end{gathered} \tag{4.10}$$

where $\varphi^*(b)$ is the principal part of the function $\varphi(b)$ as $b\to\infty$. It is evident by the method of constructing the function $\varphi(b)$ which has been expounded above that the value $\varphi^*(b_1)$ should agree asymptotically with the corresponding value of the interior solution determined by one of the forms (3.8) or (3.9).

Finally let us outline specifically the practical limits of applicability of the asymptotic equality (4.9). In order that there be no mutual influence between points of the contour L asymptotically for small μ, it is evidently sufficient that the inner and outer boundary layers be packed within the inner $\Omega-\Omega_\varepsilon$ and outer $\Omega^*-\Omega_\varepsilon^*$ domains, respectively

Taking account of the above, we obtain the condition

$$\mu < \varepsilon [\sup (H_1, H_2)]^{-1}, \qquad \varepsilon < 1 \tag{4.11}$$

approximately defining the desired boundary.

In the case of a convex domain of contact Ω, the necessity to introduce an outer boundary layer is removed, hence, condition (4.11) takes on in the case the simpler form

$$\mu < \varepsilon H_1^{-1}, \qquad \varepsilon < 1 \tag{4.12}$$

Let us summarize. A function φ (β) satisfying the integral equation (4.6) yields the principal part of the asymptotic of the solution of integral equation (1.1) in the domain $\Omega - \Omega_\varepsilon$ for small μ. Formula (3.8) or (3.9) yields the solution of integral equation (1.1) for small μ in the domain Ω_ε. Upon compliance with (4.11) they merge asymptotically on the boundary of the domain Ω_ε as is clear from the scheme for constructing the mentioned solutions, and will be verified below in examples. Therefore, an approximate solution of the integral equation (1.1) for small μ can be obtained in the whole domain Ω with the required singularity of the form $R^{-1/2}$ on the contour L.

The technique of computing contact pressures at a given point $A \in \Omega$ reduces to the following. If $A \in \Omega_\varepsilon$ this computation is carried out by means of the interior solution, if $A \in \Omega - \Omega_\varepsilon$, then it is necessary to drop a perpendicular to the contour L from this point, to determine the length of the normal n, and then to compute the contact pressures in conformity with (4.9).

5. Examples. Approximate formula to determine the force P acting on the stamp. Let us utilize the Koiter idea of approximate factorization to obtain practically acceptable solutions of the Wiener-Hopf integral equation (4.6), (4.7).

Let us approximate the function $L(u)$ defined by one of the formulas (1.3) by the expression

$$L^*(u) = u\frac{\sqrt{u^2+B^2}}{u^2+C} \tag{5.1}$$

and taking (5.1) into account, we obtain a solution of (4.6), (4.7), and also find $g(b)$ in conformity with (4.8).

Omitting traditional computations, accompanying the Wiener-Hopf method [8], let us present the final expressions for $\varphi(b)$ and $g(b)$:

(1) $f(b) \equiv \gamma$

$$\varphi(b) = \frac{\Delta\gamma}{Dh}\left(\operatorname{erf}\sqrt{Bb} + \sqrt{\frac{D}{\pi b}}\, e^{-Bb}\right), \quad 0 \leqslant b < \infty, \quad D = BC^{-1} \tag{5.2}$$

$$g(b) = \gamma\,[1 - \operatorname{erf}\sqrt{-Bb} + \sqrt{1-k}\, e^{\sqrt{Cb}} \operatorname{erf}\sqrt{-Bb(1-k)}],$$
$$-\infty < b < 0, \quad k = \sqrt{C}B^{-1}$$

(2) $f(b) = \gamma b$

$$\varphi(b) = \frac{\Delta\gamma}{h}\left[\frac{b}{D}\operatorname{erf}\sqrt{Bb} - \frac{1}{\sqrt{\pi Bb}}\, e^{-Bb}\left(1 - \frac{\sqrt{C}}{2B} - \frac{b}{D}\right)\right], \quad 0 < b < \infty \tag{5.3}$$

(3) $f(b) = \gamma b^2$

$$\varphi(b) = \frac{\Delta\gamma}{h}\left[\operatorname{erf}\sqrt{Bb}\left(\frac{b^2}{D} + \frac{C}{B^2} - \frac{2}{B}\right) - \frac{1}{\sqrt{\pi Bb}}e^{-Bb}\left(\frac{1}{B} - \frac{3\sqrt{C}}{4B^2} + \frac{Cb}{2B^2} - \frac{b^2}{D}\right)\right],$$

$$0 \leqslant b < \infty \tag{5.4}$$

The expressions for the function $g(b)$ are not needed later for cases (2) and (3).

In order to be able to carry out specific calculations, let us indicate the values of the constants B and C in the approximation (5.1) for both considered problems. Let us select these constants in such a manner that

$$\lim\left[\frac{u}{L(u)} - \frac{u}{L^*(u)}\right] = 0, \quad \lim\frac{d^2}{du^2}\left[\frac{u}{L(u)} - \frac{u}{L^*(u)}\right] = 0 \quad \text{for } u \to 0 \tag{5.5}$$

As is seen from (3.11) and (3.12), this is necessary for the correct matching between the boundary layer solution in the domain $\Omega - \Omega_\varepsilon$ and the interior solution in the domain Ω_ε (see below apropos of this). We therefore obtain:

For the first problem

$$B = 1, \qquad C = 2 \tag{5.6}$$

For the second problem with

$$\nu = 0.3$$

$$B = 1.037, \qquad C = 2.540 \tag{5.7}$$

Now let us determine the boundary layer thicknesses for cases (5.2) - (5.4) by means of (4.10). For the problem (a) we will have

$$1)\ H_1 = 0.6 \quad H_2 = 1.1, \qquad 2)\ H_1 = 0.9, \quad 3)\ H_1 = 1.3 \tag{5.8}$$

and for problem (b)

$$1)\ H_1 = 0.9 \quad H_2 = 0.8, \qquad 2)\ H_1 = 1.1, \qquad 3)\ H_1 = 1.3 \tag{5.9}$$

It is interesting to note that there is a zone of negative values for the function $g(b)$ for the case $f(b) \equiv \gamma$. This indicates that slight buckling of the surface of the layer near the stamp boundary occurs upon impression of a stamp on a layer of slight relative thickness. Such buckling is not observed in contact problems for an elastic half-space.

Let us turn to a study of a plane stamp of arbitrary planform $(\delta(Q) \equiv \delta)$. As has been shown in Sect. 3, the interior solution for this case has the form (see 3.11))

$$q(Q) = \frac{\Delta\delta}{Ah} + O(e^{-d/\mu}), \qquad Q \in \Omega_\varepsilon \tag{5.10}$$

where for $\varepsilon = 0.(9)$ the constant d equals, respectively for the considered cases

$$\text{(a)}\ 3.142, \qquad \text{(b)}\ 0.877 \quad (\nu = 0.3) \tag{5.11}$$

The boundary-layer type solution in the domain $\Omega - \Omega_\varepsilon$ is given by the first relationship in (5.2). Taking account of (5.5), it becomes on the domain boundary Ω_ε

$$q(Q) = \frac{\Delta\delta}{Ah} + O(e^{-d^*/\mu}) \tag{5.12}$$

where for $\varepsilon = 0.(9)$ the constant d^* equals, respectively, for the considered problems

$$\text{(a)}\quad 1.000, \qquad \text{(b)}\quad 1.037 \qquad (\nu = 0.3) \tag{5.13}$$

It is seen from a comparison between (5.10) and (5.12) that the interior solution and the boundary-layer type solution match asymptotically for small μ on the domain boundary Ω_ε thereby assuring an approximate determination of the contact pressures in the whole domain Ω in the complex.

We find the approximate boundary of utilization of such a complex by means of (4.11) and (4.12). Namely, taking account of (5.8) and (5.9), we obtain $\mu < 0.91$ for a stamp of arbitrary planform, $\mu < 1.67$ for a stamp of convex planform in the case of problem (a), and $\mu < 1.11$ for a stamp of arbitrary or convex planform in the case of problem (b), where $\varepsilon = 0.(9)$ was used in the calculations.

Let us examine the case of a plane stamp of elliptic planform in more detail (Ω is an ellipse with semi-axes a and b, $a > b$).

Taking into account that the minimal radius of curvature of the ellipse is $a(1 - e^2)$, where e is the eccentricity, we obtain the following relation between the parameters λ, and μ:

$$\mu = \lambda(1 - e^2)^{-1} \tag{5.14}$$

Taking into account that $n = b - y$ for $x = 0$, and $n = a - x$ for $y = 0$, on the basis of the first formula in (5.2) and of (5.10) we obtain the following approximate relationships to compute the contact pressures on the axes of the elliptical domain Ω:

$$\begin{aligned} q(x, 0) &= \frac{\Delta\delta}{Ah}\left\{\operatorname{erf}\left(\frac{B(a-x)}{h}\right)^{1/2} + \left(\frac{Ah}{\pi(a-x)}\right)^{1/2} \exp\left[-\frac{B(a-x)}{h}\right]\right\} \\ q(0, y) &= \frac{\Delta\delta}{Ah}\left\{\operatorname{erf}\left(\frac{B(b-y)}{h}\right)^{1/2} + \left(\frac{Ah}{\pi(b-y)}\right)^{1/2} \exp\left[-\frac{B(b-y)}{h}\right]\right\} \end{aligned} \tag{5.15}$$

For a plane stamp of circular planform we will have

$$q(r) = \frac{\Delta\delta}{Ah}\left\{\operatorname{erf}\left(\frac{B(a-r)}{h}\right)^{1/2} + \left(\frac{Ah}{\pi(a-r)}\right)^{1/2} \exp\left[-\frac{B(a-r)}{h}\right]\right\}, \quad r = \sqrt{x^2 + y^2} \tag{5.16}$$

From the statics conditions for the stamp we now find the connection between the force P acting on the plane circular stamp, and its shrinkage δ

$$\begin{aligned} P = \frac{\pi\Delta\delta a^2}{Ah}\Bigg[&\operatorname{erf}\left(\frac{B}{\lambda}\right)^{1/2}\left(1 - \frac{\lambda}{B} + \frac{3\lambda^2}{4B^2} + 2\lambda\sqrt{\frac{A}{B}} - \lambda^2\frac{\sqrt{A}}{B^{3/2}}\right) + \\ &+ \left(\frac{\lambda}{\pi B}\right)^{1/2} e^{-B/\lambda}\left(1 - \frac{3\lambda}{2B} + 2\lambda\left(\frac{A}{B}\right)^{1/2}\right)\Bigg] \end{aligned} \tag{5.17}$$

Let us note that it is difficult to obtain an analogous formula for the force P in the case of a plane stamp of arbitrary (particularly elliptical) planform because of the lack of a single analytical expression for the contact pressures in the whole contact domain Ω analogous to (5.16). Consequently, we elucidate below an approximate method for determining the force P acting, for small μ, on a plane stamp of arbitrary planform.

Taking account of (5.10) and the first formula in (5.2), let us represent the asymptotic solution, for small μ, for a plane stamp of arbitrary planform as

$$q(Q) \sim \frac{\Delta\delta}{Ah}\left[1 + \begin{cases} 0, & Q \in \Omega_\varepsilon \\ \operatorname{erf}\sqrt{Bn/h} - 1 + \sqrt{Ah/\pi n}\exp(-Bn/h), & Q \in \Omega - \Omega_\varepsilon \end{cases}\right] \tag{5.18}$$

We now obtain the following approximate relationship for the force P acting on the stamp:

$$P = \frac{\Delta\delta}{Ah}\left\{S + lh\int_0^{1/\mu}\left[\operatorname{erf}\sqrt{Bb} - 1 + \sqrt{\frac{A}{\pi b}}\exp(-Bb)\right]db\right\} \tag{5.19}$$

where S is the area of the domain Ω, l the perimeter of the contour L, and ε is taken equal to one. Letting the upper limit of the integral in (5.19) be infinity asymptotically for small μ, and evaluating the integral, we finally obtain

$$P = \frac{\Delta\delta}{Ah}\left[S + hl\left(\sqrt{\frac{A}{B}} - \frac{1}{2B}\right)\right] = \frac{\Delta\delta}{Ah}(S + phl) \tag{5.20}$$

Here the constant p for the considered problems is, respectively

$$\text{(a)}\quad 0.2071, \qquad \text{(b)}\quad 0.1453 \tag{5.21}$$

It is easy to note that the approximate formula (5.20) yields the first two members of the asymptotic of (5.17) for small λ for the case of a circular stamp. Results are given in the table for a numerical comparison between (5.20) and (5.17) for the case of a circular stamp. For convenience, we have used the notation $\varkappa = P(4\Delta\delta a)^{-1}$. Presented in the last column of the Table are appropriate values of the quantity $\varkappa$ obtained in [9] by completely different means

Table

		(5.17)		(5.20)		[9]	
λ		1	1/2	1	1/2	1	1/2
(a)	$\varkappa$	2.23	3.81	2.22	3.79	2.20	3.72
(б)	$\varkappa$	2.57	4.48	2.48	4.41	2.48	4.34

The approximate formula (5.20) becomes for the case of a stamp of elliptical planform:

$$P = \frac{\Delta\delta a}{A\lambda}[\pi\sqrt{1-e^2} + 4p\lambda E(e)] \tag{5.22}$$

where $E(e)$ is the complete elliptic integral of the second kind.
Let us study the case of a parabolic stamp of circular planform

$$\delta(r) = \delta - \gamma r^2/a^2 = (\delta - \gamma) + 2\gamma\lambda b - \gamma\lambda^2 b^2 \tag{5.23}$$

On the basis of (3.11) and (3.12) the interior solution for this case is

$$q(r) \sim \frac{\Delta}{Ah}\left[\delta - \gamma\left(\frac{r^2}{a^2} - 4D_1\lambda^2\right)\right], \qquad r \in \Omega_\varepsilon \tag{5.24}$$

We form the solution of boundary-layer type in the domain $\Omega - \Omega_\varepsilon$ by means of the first formula in (5.2), and of (5.3), (5.4), (5.23). We have

$$\begin{aligned} q(r) = {} & \frac{\Delta}{Ah}\operatorname{erf}\left(\frac{B(a-r)}{h}\right)^{1/2}\left[\delta - \gamma\left(\frac{r^2}{a^2} + \frac{\lambda^2}{B^2} - \frac{2A\lambda^2}{B}\right)\right] + \\ & + \frac{\Delta}{\sqrt{\pi Ah(a-r)}}\exp\left(-B\frac{a-r}{h}\right)\left\{\delta - \gamma\left[1 - \frac{1}{\sqrt{AB}}\left(1-\frac{r^2}{a^2}\right) - \right.\right. \\ & \left.\left. - \frac{\lambda}{2\sqrt{A}\,B^{3/2}}\left(1-\frac{r}{a}\right) + \lambda\left(2\sqrt{\frac{A}{B}} - \frac{1}{B}\right) + \lambda^2\left(\frac{3}{4B^2} - \frac{\sqrt{A}}{B^{3/2}}\right)\right]\right\}, \qquad r \in \Omega - \Omega_\varepsilon \end{aligned} \tag{5.25}$$

As has already been remarked above, compliance with condition (5.5) assures the correct asymptotic matching between the interior and boundary layer solutions based on the approximation (5.1). Indeed, if the relationship

$$\frac{A}{2B} - \frac{1}{4B^2} = D_1 \tag{5.26}$$

whose validity results from (5.5), is taken into account, then the principal term in the asymptotic (5.25) for small λ will coincide with (5.24) on the boundary Ω_ε, as is easily seen.

It hence follows that the asymptotic solution, for small λ, of the problem for a circular parabolic stamp can be represented by a single analytical expression in the whole contact domain Ω, in the form of (5.25).

For the parabolic stamp case we now obtain an asymptotic solution for small λ which vanishes on the contour of the domain Ω, i.e. for $r = a$. On the basis of (5.25), we find

$$\begin{aligned} q(r) = \frac{\Delta\gamma}{Ah}\Bigg\{ & \operatorname{erf}\left(\frac{B(a-r)}{h}\right)^{1/2}\left[1 - \frac{r^2}{a^2} + \lambda\left(1 + \lambda\left(\frac{A}{B}\right)^{1/2}\right)\left(2\left(\frac{A}{B}\right)^{1/2} - \frac{1}{B}\right) - \frac{\lambda^2}{4B^2}\right] + \\ & + \left(\frac{\lambda(a-r)}{\pi aB}\right)^{1/2}\exp\left(-B\frac{a-r}{h}\right)\left(1 + \frac{r}{a} + \frac{\lambda}{2B}\right)\Bigg\} \end{aligned} \tag{5.27}$$

under the condition

$$\delta = \gamma\left(1 + 2\lambda\left(\frac{A}{B}\right)^{1/2} - \frac{\lambda}{B} + \frac{3\lambda^2}{4B^2} - \frac{\lambda^2\sqrt{A}}{B^{3/2}}\right) \tag{5.28}$$

which can be utilized to determine the radius a of the contact domain Ω for given δ and γ.

For the force P acting on a parabolic stamp we obtain

$$\begin{aligned} P = \frac{2\pi\Delta\gamma a}{A}\Bigg\{ & \lambda\left(\frac{A}{B}\right)^{1/2}\left(1 - \frac{\lambda}{2B} + \lambda\left(\frac{A}{B}\right)^{1/2}\right)\left[\operatorname{erf}\left(\frac{B}{\lambda}\right)^{1/2}\left(\frac{1}{\lambda} - \frac{1}{B} + \frac{3\lambda}{4B^2}\right) + \right. \\ & \left. + \frac{e^{-B/\lambda}}{\sqrt{\pi B\lambda}}\left(1 - \frac{3\lambda}{2B}\right)\right] + \operatorname{erf}\sqrt{\frac{B}{\lambda}}\left(\frac{1}{4\lambda} - \frac{1}{2B} + \frac{5\lambda}{8B^2} - \frac{3\lambda^2}{8B^3} - \frac{15\lambda^3}{64B^4}\right) + \\ & + \frac{e^{-B/\lambda}}{\sqrt{\pi B\lambda}}\left(\frac{1}{4} - \frac{5\lambda}{8B} + \frac{17\lambda^2}{16B^2} + \frac{15\lambda^2}{32B^3}\right)\Bigg\} \end{aligned} \tag{5.29}$$

We determine the limits of applicability of (5.25), (5.27) - (5.29) in λ in conformity with (4.12) and (5.8), (5.9). For $\varepsilon = 0.(9)$ we will have $\lambda < 0.77$ for both problems.

Finally, we obtain the asymptotic solution for small λ, for an oblique circular stamp. Let $\delta(Q) = \alpha r a^{-1} \cos\varphi$ then as is known, $q(Q)=q_1(r)\cos\varphi$ where r and φ are polar coordinates. The function $q_1(r)$ can be obtained by differentiation of the right side of (5.27) ([10], Sect. 1) with respect to r. It is here necessary only to replace $-2\gamma a^{-1}$ by α. We thereby find

$$q_1(r) = \frac{\Delta\alpha}{Ah}\left\{\frac{r}{a}\operatorname{erf}\left(\frac{B(a-r)}{h}\right)^{1/2} + \right.$$
$$\left. + \left(\frac{h}{\pi B(a-r)}\right)^{1/2}\exp\left(-B\frac{a-r}{h}\right)\left[\sqrt{AB}\left(1-\frac{\lambda}{2B}+\lambda\left(\frac{A}{B}\right)^{1/2}\right)-1+\frac{r}{a}\right]\right\} \qquad (5.30)$$

The value of the moment M applied to an oblique circular stamp coincides, as can be shown, with the value of the force P defined by (5.29) to the accuracy of a sign. It is hence necessary just to replace $-2\gamma a^{-1}$ by α.

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